Patterns of ideals of numerical semigroups

Klara Stokes
School of Engineering Sciences
University of Skövde, Box 408
54128 Skövde Sweden
klara.stokes@his.se

Abstract

This article introduces patterns of ideals of numerical semigroups, thereby unifying previous definitions of patterns of numerical semigroups. Several results of general interest are proved. More precisely, this article presents results on the structure of the image of patterns of ideals, and also on the structure of the sets of patterns admitted by a given ideal.

1 Introduction

A *numerical semigroup* S is a subset of the non-negative integers (denoted by $\mathbb{Z}_+$) that contains zero, is closed under addition and has finite complement in $\mathbb{Z}_+$. The set of non-zero elements in S is denoted by $M(S)$.

Elements in the complement $\mathbb{Z}_+ \setminus S$ are called *gaps*, and the number of gaps is the *genus* of S. The smallest element of $M(S)$ is the *multiplicity* of S and it is denoted by $m(S)$. The largest integer not in S is the *Frobenius element* and is denoted by $F(S)$. The number $F(S)+1$ is called the *conductor* of S and is denoted by $c(S)$. An integer $x \notin S$ is *pseudo-Frobenius* if $x + s \in S$ for all $s \in M(S)$. The set of pseudo-Frobenius integers is denoted by $PF(S)$. Note that $F(S) \in PF(S)$ and that $F(S)$ is the maximum of the elements in $PF(S)$.

It can be proved that, given a numerical semigroup S, there exists a unique minimal set of elements $B \subset M(S)$ such that any element in S can be expressed as a linear combination of elements from B. The elements in B are called *minimal generators* of S and they are exactly the elements of $M(S)$ that can not be obtained as the sum of two elements of $M(S)$. The cardinality of B is always finite. More precisely it is always less or equal to the multiplicity of S. A numerical semigroup has *maximal embedding dimension* if the number of minimal generators equals the multiplicity.

A *relative ideal* of a numerical semigroup S is a set $H \subseteq \mathbb{Z}$ satisfying $H+S \subseteq H$ and $H + d \subseteq S$ for some $d \in S$. A relative ideal contained in S is an *ideal* of S. An ideal is *proper* if it is distinct from S. The set of proper ideals of S has a maximal element with respect to inclusion. This ideal is called the *maximal*

ideal of S, and equals $M(S)$, the set of non-zero elements of S. The *dual* of a relative ideal H is the relative ideal $H^* = (S - H) = \{z \in \mathbb{Z} : z + H \subseteq S\}$.

A *pattern* admitted by an ideal I of a numerical semigroup S is a multivariate polynomial function $p(X_1, \ldots, X_n)$ which returns an element $p(s_1, \ldots, s_n) \in S$ when evaluated on any non-increasing sequence $(s_1, \ldots, s_n)$ of elements in I. We say that the ideal I *admits* the pattern. If $I = S$, then we say that the numerical semigroup S admits the pattern. Note that a pattern admitted by an ideal I of a numerical semigroup S is also admitted by any ideal $J \subseteq I$.

We will identify the pattern with its polynomial, and, for example, say that the pattern is linear and homogeneous, when the pattern polynomial is linear and homogeneous. The length of a pattern is the number of indeterminates and its degree is the degree of the pattern polynomial. One pattern p *induces* another pattern q if any ideal of a numerical semigroup that admits p also admits q. Two patterns are equivalent if they induce each other. If any ideal satisfying a given condition c that admits p also admits q, then we say that p induces q under the condition c. Two patterns are equivalent under the condition c if they induce each other under the condition c.

Homogeneous linear patterns admitted by numerical semigroups were introduced by Bras-Amorós and García-Sánchez in [4]. The patterns that they considered were all defined by homogeneous linear multivariate polynomials with the whole numerical semigroup as domain. Examples of homogeneous patterns are the homogeneous linear patterns with positive coefficients. It is easy to see that these patterns are admitted by any numerical semigroup. Arf numerical semigroup are characterized by admitting the homogeneous linear "Arf pattern" $X_1 + X_2 - X_3$. Homogeneous linear patterns of the form $X_1 + \cdots + X_k - X_{k+1}$ generalise the Arf pattern and are called *subtraction patterns* [4].

The definition of pattern from S to S does not allow for non-homogeneous patterns with constant term outside S. To overcome this problem, when the non-homogeneous patterns were introduced in [5], it was with $M(S)$ as domain. Note that with this definition $X + a$ with $a \in PF(S)$ is a non-homogeneous linear pattern admitted by S. Admitting the non-homogeneous linear pattern $X_1 + X_2 - m(S)$ is equivalent to the property of maximal embedding dimension. Since this pattern is induced by the pattern $X_1 + X_2 - X_3$, this implies that a numerical semigroup that is Arf is always of maximal embedding dimension.

Further examples of non-homogeneous linear patterns can be found in the numerical semigroups associated to the existence of combinatorial configurations (see [6]). It was proved in [13, 14] that such numerical semigroups admit the patterns $X_1 + X_2 - n$ for $n \in \{1, \ldots, \gcd(r, k)\}$ and $X_1 + \cdots + X_{rk/\gcd(r,k)} + 1$, where r and k are positive integers that depend on the parameters of the combinatorial configuration. This example motivates the study of a set of patterns that are admitted simultaneously by the same numerical semigroup.

Another example of a non-homogeneous linear pattern is $qX_1 - qm(S)$, which is admitted by a Weierstrass semigroup S of multiplicity $m(S)$ of a rational place of a function field over a finite field of cardinality q, for which the Geil-Matsumoto bound and the Lewittes' bound coincide [3]. Similarly, the pattern $(q-1)X_1 - (q-1)m(S)$ is admitted if and only if the Beelen-Ruano's bound

equals $1 + (q-1)m$ [5].

Patterns can be used to explore the properties of the numerical semigroup admitting them. For example, the calculations of the formulae for the notable elements of Mersenne numerical semigroups in [10] rely on the fact that all Mersenne numerical semigroups generated by a consecutive sequence of Mersenne numbers admit the non-homogeneous pattern $2X_1 + 1$. Similarly, the non-homogeneous patterns admitted by numerical semigroups associated to the existence of combinatorial configurations were used to improve the bounds on the conductor of these numerical semigroups.

In this article we study patterns of ideals of numerical semigroups.

Section 2 contains basic results about the properties of the image of patterns. For example, it is proved that if the greatest common divisor of the coefficients of the pattern p is one and I is an ideal of a numerical semigroup S, then the image $p(I)$ of a pattern is always an ideal of a numerical semigroup. Also, sufficient conditions are given for when $p(I) \subseteq S$.

Section 3 presents an upper bound of the smallest element c in $p(I)$ such that all integers larger than c belong to $p(I)$, under the condition that the greatest common divisor of the coefficients of p is one. By dividing p by the greatest common divisor of its coefficients, this result makes it possible to calculate $p(I)$ for any admissible pattern p.

Section 4 introduces the concepts endopattern and surjective pattern of an ideal, and gives some sufficient and necessary conditions on patterns to have these properties.

In Section 5, we generalize the notion of closure of a numerical semigroup with respect to a homogeneous linear pattern to the closure of an ideal of a numerical semigroup with respect to a non-homogeneous linear pattern. We also prove a necessary condition for when the closure of an ideal with respect to a non-homogeneous pattern can be calculated by repeatedly applying the pattern.

In Section 6 we prove that the set of patterns admitted by an ideal of a numerical semigroup has the structure of a semigroup, semiring or semiring algebra, depending on if the length and the degree of the patterns is fixed.

Section 7 introduces a generalization of pseudo-Frobenius as a useful tool in the analysis of the structures defined in Section 6.

Section 8 introduces infinite chains of ideals of numerical semigroups where the subsequent ideal I_i is the image of the preceding ideal I_{i-1} under a pattern p which is admitted by the first pattern in the chain, and hence by them all.

Finally, Section 9 defines polynomial composition of patterns, hence providing yet another operation that creates a pattern admitted by an ideal from several patterns admitted by that ideal.

2 The image of a pattern

A pattern p admitted by an ideal I of a numerical semigroup S returns elements in S when evaluated over the non-increasing sequences of elements of I. We will

now study the image $p(I)$ of I under p.

We will need the following well-known result.

Lemma 1. *Let $A \subseteq \mathbb{Z}_+$ be closed under addition. Then A does not have finite complement in $\mathbb{Z}_+$ if and only if $A \subseteq u\mathbb{Z}$ for some $u > 1$.*

Proof. If A does not have finite complement, then A does not contain x, y such that $\gcd(x, y) = 1$, since otherwise the maximal ideal of the numerical semigroup $\langle x, y\rangle$, which has finite complement, would be contained in A. Therefore $\gcd(A) = u$ for some $u > 1$ so that $A \subseteq u\mathbb{Z}$.

If $u\mathbb{Z}$ is an ideal of $\mathbb{Z}$ with $u > 1$ and $A \subseteq u\mathbb{Z}$, then clearly $|\mathbb{Z}_+ \setminus A|$ is infinite. □

Lemma 2. *If I is an ideal of some numerical semigroup S, then there is a $c \in I$ such that $z \in I$ for all $z \in \mathbb{Z}$ with $z \geq c$.*

Proof. If I is an ideal of some numerical semigroup S, then $I + S \subseteq I$, implying that $|\mathbb{Z}_+ \setminus I| < \infty$. Therefore there is a $c \in I$ such that $z \in I$ for all $z \in \mathbb{Z}$ with $z \geq c$. □

If, in Lemma 2, $I = S$, then the integer c is the conductor of S. If I is a proper ideal, then we call c the *maximum of the small elements of I.*

Theorem 3. *Let $p(X_1, \dots, X_n) = a_1X_1 + \cdots + a_nX_n$ be a homogeneous linear pattern admitted by $\mathbb{Z}_+$ and let I be an ideal of a numerical semigroup S. Then $p(I)$ is an ideal of some numerical semigroup if and only if* $\gcd(a_1, \dots, a_n) = 1$.

Proof. Assume that $\gcd(a_1, \dots, a_n) = 1$ and let c be the maximum of the small elements of I (see Lemma 2). Let $u > 1$ and $s \in I \cap u\mathbb{Z}$ with $s \geq c$. Then $s + 1, s + u \in I$, with $s + 1 \notin u\mathbb{Z}$, $s + u \in u\mathbb{Z}$ and $s + u > s + 1 > s$. Since $\gcd(a_1, \dots, a_n) = 1$ there is an $i \in [1, n]$ such that a_i is not a multiple of u. Therefore $\sum_{j=1}^{i-1} a_j(s + u) + a_i(s + 1) + \sum_{j=i+1}^{n} a_j s \in p(I) \setminus u\mathbb{Z}$. Lemma 1 implies that $p(I)$ has finite complement in $\mathbb{Z}_+$.

Note that if $x_1, \dots, x_n$ and $y_1, \dots, y_n$ are non-increasing sequences of I, then so is $x_1 + y_1, \dots, x_n + y_n$. Since the pattern p is linear and homogeneous we have $p(x_1, \dots, x_n) + p(y_1, \dots, y_n) = p(x_1 + y_1, \dots, x_n + y_n) \in p(I)$ for all non-increasing sequences $x_1, \dots, x_n \in I$ and $y_1, \dots, y_n \in I$, that is, $a + b \in p(I)$ for all $a, b \in p(I)$. Hence $p(I)$ is closed under addition. (That linearity of p implies that $p(I)$ is closed under addition was first noted in [4].)

Together, the above imply that if $0 \in p(I)$, then $p(I)$ is a numerical semigroup, and if $0 \notin p(I)$, then $p(I)$ is the maximal ideal of the numerical semigroup $p(I) \cup \{0\}$. In any case, $p(I)$ is an ideal of a numerical semigroup.

Now assume that $\gcd(a_1, \dots, a_n) = u > 1$. Then clearly $p(I) \subseteq u\mathbb{Z}$, so that $p(I)$ does not have finite complement in $\mathbb{Z}_+$ and can not be the ideal of any numerical semigroup. □

Note that in Theorem 3, either $p(I)$ is a numerical semigroup, or $p(I)$ is the maximal ideal of the numerical semigroup $p(I) \cup \{0\}$, depending on whether $0 \in p(I)$ or not. When I is a proper ideal, then $0 \in p(I)$ exactly when $\sum_{i=1}^{n} a_i = 0$ (see Proposition 18). When $I = S$, then Theorem 3 implies the following result.

Corollary 4. *Let $p(X_1, \ldots, X_n) = a_1X_1 + \cdots + a_nX_n$ be a homogeneous linear pattern admitted by $\mathbb{Z}_+$ and let S be a numerical semigroup. Then $p(S)$ is a numerical semigroup if and only if* $\gcd(a_1, \ldots, a_n) = 1$.

Proof. Apply Theorem 3 with $I = S$ and note that $p(0, \ldots, 0) = 0 \in p(S)$. $\square$

Clearly the numerical semigroup $p(S)$ is contained in the original numerical semigroup S if and only if p is admitted by S.

Following [4], a linear homogeneous pattern $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_iX_i$ is *premonic* if $\sum_{i=1}^{n'} a_i = 1$ for some $n' \leq n$. We say that a linear non-homogeneous pattern $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_iX_i + a_0$ is premonic if $p(X_1, \ldots, X_n) - a_0$ is premonic. If $a_1 = 1$ then p is *monic* and so all monic patterns are premonic.

Lemma 5. *If p is a premonic linear homogeneous pattern admitted by a numerical semigroup S, then $p(S) = S$.*

Proof. If p is a linear homogeneous pattern admitted by S and p is premonic, then $\sum_{i=1}^{n'} a_is + \sum_{j=n'+1}^{n} a_j0 = s$ for all $s \in S$, so that $S \subseteq p(S)$. Clearly $p(S) \subseteq S$, so that $p(S) = S$. $\square$

Moreover, the image of a premonic linear pattern, homogeneous or not, admitted by an ideal I of a numerical semigroup S, is an ideal of S.

Lemma 6. *Let p be a premonic linear pattern admitted by an ideal I of a numerical semigroup S. Then $p(I)$ is an ideal of S.*

Proof. Clearly, if p is a pattern admitted by I then $p(I) \subseteq S$. If $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_iX_i + a_0$ is a premonic pattern then $\sum_{i=1}^{n'} a_i = 1$ for some $1 \leq n' \leq n$. If $s_1, \ldots, s_n$ is a non-increasing sequence of elements from I and $s \in S$ then $s_1 + s, \ldots, s_{n'} + s, s_{n'+1}, \ldots, s_n$ is a non-increasing sequence of elements from I for any $1 \leq n' \leq n$. We have $p(s_1, \ldots, s_n) + s = \sum_{i=1}^{n} a_is_i + a_0 + s = \sum_{i=1}^{n} a_is_1 + a_0 + (\sum_{i=1}^{n'} a_i)s = \sum_{i=1}^{n'} a_i(s_i + s) + \sum_{i=n'+1}^{n} a_is_i + a_0 = p(s_1 + s, \ldots, s_{n'} + s, s_{n'+1}, \ldots, s_n) \in p(I)$ for all non-increasing sequences $s_1, \ldots, s_n$ of elements from I and for all $s \in S$. Therefore $p(I) + S \subseteq p(I)$, and $p(I)$ is an ideal of S. $\square$

Note that if $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_iX_i$ is a premonic linear pattern admitted by the maximal ideal $M(S)$ of a numerical semigroups S, then $\gcd(a_1, \ldots, a_n) = 1$, so that, by Theorem 3, $p(M(S))$ is either a numerical semigroup contained in S or the maximal ideal of a numerical semigroup contained in S. But although an ideal of S that contains zero must be equal to S, it is not true in general that if p is a premonic, homogeneous linear pattern admitted by $M(S)$, then $p(M(S)) \cup \{0\} = S$. Consider for example the pattern $X_1 + X_2$ with image $2M(S) \subsetneq M(S)$. However, as we have already seen, if p is a premonic linear homogeneous pattern admitted by a numerical semigroup S, then $p(S) = S$.

Example 7. *The Arf pattern $p(X_1, X_2, X_3) = X_1 + X_2 - X_3$ is a monic linear homogeneous pattern. If S is a numerical semigroup Arf, then $p(S) = S$, and*

since $p(0,0,0) = 0$ *and* $p^{-1}(0) = (0,0,0)$ *also* $p(M(S)) = M(S)$. *If* I *is an ideal of* S, *then* $I \subseteq p(I)$, *but in general it is not true that* $p(I) \subseteq I$. *For example, if* $S = \langle 3,5,7 \rangle$ *and* $I = S \setminus \{0,7\}$, *then* $p(5,5,3) = 7 \in p(I) \setminus I$ *and* $p(I) = M(S)$.

Finally, we show that there is a relation between relative ideals and premonic linear non-homogeneous patterns.

Lemma 8. *Let* $p(X_1,\dots,X_n) = a_1X_1 + \cdots + a_nX_n + a_0$ *be a linear non-homogeneous pattern admitted by an ideal* I *of a numerical semigroup* S *and let* $q(X_1,\dots,X_n) = p(X_1,\dots,X_n) - a_0$ *be the homogeneous linear part of* p. *If* p *is premonic (and therefore also* q*), then* $q(I)$ *is a relative ideal of* S.

Proof. Since $p(X_1,\dots,X_n) = q(X_1,\dots,X_n) + a_0$ by Lemma 6 we have $q(I) + S \subseteq q(I)$ and $q(I) + a_0 \subseteq S$. $\square$

3 Calculating the image of a pattern

The following result is useful for calculating the image $p(I)$ of an admissible linear homogeneous pattern.

Lemma 9. *Let* I *be an ideal of a numerical semigroup. Let* $p(X_1,\dots,X_n) = \sum_{i=1}^{n} a_iX_i$ *be a homogeneous linear pattern with* $\gcd(a_1,\dots,a_n) = d(\geq 1)$ *and let* $b_1,\dots,b_n$ *be (non-unique) integers such that* $a_1b_1 + \cdots + a_nb_n = d$. *It is not assumed that* p *is admitted by* I. *Then* $p(s_1,\dots,s_n) + d \in p(I)$ *for all non-increasing sequences* $s_1,\dots,s_n \in I$ *such that* $s_1 + b_1,\dots,s_n + b_n$ *is also a non-increasing sequence of elements from* I.

Proof. We have $p(s_1,\dots,s_n)+d = p(s_1,\dots,s_n)+p(b_1,\dots,b_n) = p(s_1+b_1,\dots,s_n+b_n)$. Therefore, if $s_1 + b_1,\dots,s_n + b_n$ is a non-increasing sequence of elements from I, then $p(s_1,\dots,s_n) + d \in p(I)$. $\square$

Note that any choice of $b_1,\dots,b_n$ such that $a_1b_1 + \cdots + a_nb_n = d$ will do. In practice it may be useful to instead require that $s_1 \geq \cdots \geq s_n \geq c(I)$ and $s_1 + b_1 \geq \cdots \geq s_n + b_n \geq c(I)$ where $c(I)$ is the smallest element in I such that $z \in I$ for all integers $z \geq c$. Clearly then both $s_1,\dots,s_n$ and $s_1 + b_1,\dots,s_n + b_n$ are non-increasing sequences of elements from I.

Theorem 10. *Let* I, $c(I)$, p, d *and* $b_1,\dots,b_n$ *be as in Lemma 9. Then* $J = p(I)/d$ *is an ideal of a numerical semigroup. Let* $c(J)$ *be the maximum of the small elements of* J. *Also let* $\alpha = \sum_{i=1}^{n} a_i/d$. *Then* $c(J) < p(s_1,\dots,s_n)/d$ *whenever* $s_n \geq c(I) - \min(0,(\alpha-1)b_n)$ *and* $s_i \geq s_j + \max(0,(\alpha-1)(b_j - b_i))$ *for* $1 \leq i < n$.

Proof. If $d = \gcd(a_1,\dots,a_n)$ then d divides all elements of $p(I)$. Dividing $p(I)$ by d gives the image of I under the pattern $q = p/d = \sum_{i=1}^{n} \frac{a_i}{d} X_i$ which has relatively prime coefficients $c_i = \frac{a_i}{d}$ such that $c_1b_1 + \cdots + c_nb_n = 1$ and $\sum_{i=1}^{n} c_i = \alpha$. Therefore, by Theorem 3, $J = q(I)$ is an ideal of some semigroup.

Any non-increasing sequence $s_1,\dots,s_n \in \mathbb{Z}$ with $s_n \geq c(I)$ is a non-increasing sequence of elements of I. Note that the b_i:s can be negative integers. Take

$s_n \geq c(I) - \min(0, (\alpha-1)b_n)$ and $s_i \geq s_j + \max(0, (\alpha-1)(b_j - b_i))$ for $1 \leq i < n$. Under these conditions we have that $s_i + tb_i \geq s_j + tb_j \geq c(I)$ for all $1 \leq i \leq j \leq n$ and $0 \leq t \leq \alpha - 1$, so that $q(s_1 + tb_1, \ldots, s_n + tb_n) \in q(I)$. Now note that $q(s_1+x+(t+1)b_1, \ldots, s_n+x+(t+1)b_n) = q(s_1+x+tb_1, \ldots, s_n+x+tb_n)+1$ for all $0 \leq t \leq \alpha-1$ and for all $x \geq 0$. Also, $q(s_1+x, \ldots, s_n+x) = q(s_1, \ldots, s_n)+\alpha x$. Therefore $q(I)$ contains all integers larger or equal to $q(s_1, \ldots, s_n)$ with $s_n \geq c(I) - \min(0, (\alpha-1)b_n)$ and $s_i \geq s_j + \max(0, (\alpha-1)(b_j - b_i))$ for $1 \leq i < n$. $\square$

Theorem 10 implies that the set of non-increasing sequences of I which is needed for calculating explicitly $p(I)$ is finite. However, in practice this number will depend on the choice of $b_1, \ldots, b_n$.

A linear pattern $p(X_1, \ldots, X_n) = \sum_{i=1}^n a_i X_i + a_0$ is called *strongly admissible* if the partial sums $\sum_{i=1}^{n'} a_i \geq 1$ for all $1 \leq n' \leq n$. (Strongly admissible patterns were introduced differently in [4], but the two definitions are equivalent).

Lemma 11. *Let C be a positive integer constant, and let $p(X_1, \ldots, X_n) = \sum_{i=1}^n a_i X_i + a_0$ be a strongly admissible linear pattern. Let $Y_t = \{(t, s_2, \ldots, s_n) : \forall\ 2 \leq i \leq n\ s_i \in I, \forall\ 2 \leq i \leq n-1\ s_i \geq s_{i+1}\}$ and let $Y(C) = \bigcup_{t \leq x,\ t \in I} Y_t$ for the smallest $x \in I$ such that $p(x, s_2, \ldots, s_n) \geq C$ for all $s_2, \ldots, s_n \in I$ such that $x \geq s_1 \geq \cdots \geq s_n$. Then $Y(C)$ is a well-defined finite set and contains the set of non-increasing sequences $s_1, \ldots, s_n \in I$ such that $p(s_1, \ldots, s_n) < C$.*

Proof. First we prove that if $\sum_{i=1}^{n'} a_i \geq 1$ for all $1 \leq n' \leq n$, then there is an $x \in I$ such that $p(x, s_2, \ldots, s_n) \geq C$ for all $s_2, \ldots, s_n \in I$ such that $x \geq s_2 \geq \cdots \geq s_n$. Let $s_1 \geq \cdots \geq s_n \geq 0$. Then $p(s_1, \ldots, s_n) = \sum_{i=1}^n a_i s_i + a_0 = \sum_{i=1}^{n-1} (\sum_{j=1}^i a_j)(s_i - s_{i+1}) + \sum_{j=1}^n a_j s_n + a_0 \geq \sum_{i=1}^{n-1} 1 \cdot (s_i - s_{i+1}) + 1 \cdot s_n + a_0 = s_1 + a_0$. Therefore, by taking $x = s_1 \geq C - a_0$, one has $p(x, s_2, \ldots, s_n) \geq C$ for all $s_2, \ldots, s_n \in I$ such that $x \geq s_2 \geq \cdots \geq s_n$, so Y is a well-defined finite set.

Now note that we have also proved that for all $s_1, \ldots, s_n \in I$ with $s_1 > x$, $p(s_1, \ldots, s_n) \geq C$. Consequently, if $s = (s_1, \ldots, s_n)$ is a non-increasing sequence of elements of I such that $p(s_1, \ldots, s_n) < C$, then $s \in Y$.

$\square$

The following algorithm calculates $p(I)$ by calculating first an upper bound $C \geq c(J)$ and then calculating $p(s)$ for all $s \in Y(C)$.

Lemma 12. *Let notation be as in Lemma 9 and Theorem 10. Assume also that the admissible homogeneous pattern polynomial $p(X_1, \ldots, X_n) = \sum_{i=1}^n a_i X_i$ is strongly admissible. The following algorithm can be used to calculate $p(I)$. .*

1. *Set $q = p/d$.*

2. *Calculate $C = q(s_1, \ldots, s_n)$ with $s_n = c(I) + \min(0, \alpha b_n)$ and $s_i = s_j + \min(0, \alpha(b_j - b_i))$ for $1 \leq i < n$.*

3. *Calculate $Q := \{q(s_1, \ldots, s_n) : (s_1, \ldots, s_n) \in Y(C)\}$ where $Y(C)$ is the set defined in Lemma 11.*

4. *Now $q(I) = Q \cup \{z \in \mathbb{Z} : z \geq C\}$ and $p(I) = \{ds : s \in q(I)\}$.*

Proof. By combining Theorem 10 and Lemma 11. □

Given a linear strongly admissible pattern polynomial and two ideals I of a numerical semigroup S and J of a numerical semigroup S', step 3 in the algorithm in Lemma 12 alone can be used to determine whether or not $p(I) \subseteq J$, after defining $C = \min(s \in J : n \in J \ \forall \ n \in \{z \in \mathbb{Z} : z \geq s\})$. In the particular case when I is an ideal of a numerical semigroup S and $J = S$ (and so C is the conductor of S), this calculation determines whether or not I admits p. The existence of an algorithm that determines if a strongly admissible pattern is admitted by a numerical semigroup was first announced in [4].

Lemma 13. *Let I be an ideal of a numerical semigroup, let $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i + a_0$ be an admissible linear pattern and let*

$$q(X_1, \ldots, X_n) = p(X_1, \ldots, X_n) - a_0.$$

Then $p(I) = q(I) + a_0$.

Proof. Indeed, any element in $p(I)$ is of the form $p(s_1, \ldots, s_n) = \sum_{i=1}^{n} a_i s_i + a_0 = q(s_1, \ldots, s_n) + a_0$. □

Together Lemma 12 and Lemma 13 can be used to calculate the image of an ideal of a numerical semigroup under a linear strongly admissible pattern p in a finite number of steps.

4 Patterns of ideals of numerical semigroups

In this article a pattern admitted by an ideal I of a numerical semigroup S is a multivariate polynomial function which evaluated on non-increasing sequences of elements from I returns an element of S. This definition generalises previous definitions of patterns admitted by numerical semigroups. Indeed, a homogeneous linear pattern as defined in [4] is according to our definition still a pattern admitted by a numerical semigroup. However, a non-homogeneous linear pattern as defined in [5] is now a pattern admitted by the maximal ideal of some numerical semigroup.

The concept can be generalised further, for example by relaxing the criteria that the codomain of a pattern admitted by an ideal necessarily should be a numerical semigroup containing the ideal. Then the codomain of the pattern can be another numerical semigroup, or generalising even more, an ideal of some numerical semigroup. It is possible to go even further by considering relative ideals instead of ideals. One can also restrict to patterns with some particular property like for example linearity or homogeneity.

We say that a linear pattern that returns an element in I when evaluated on the non-increasing sequences of elements of I is an *endopattern* of I. A pattern admitted by an ideal I with codomain J is *surjective* when $p(I) = J$. A surjective endopattern of I is therefore a pattern p such that $p(I) = I$. Formally,

an endopattern of I is an endomorphism of the set of non-increasing sequences of n elements from I. Note that, for example, the map $x \mapsto (x, \ldots, x)$ is an embedding of the image of the pattern in the set of non-increasing sequences of n elements from I.

In this article, the focus is on linear endopatterns of numerical semigroups and ideals of numerical semigroups, in particular maximal ideal. To avoid confusion we will each time explicitly state the properties of the patterns that we consider in each moment.

We start with necessary conditions for linear patterns to be endopatterns and surjective endopatterns of numerical semigroups. We will repeatedly make use of the following result, first proved in [4] for homogeneous linear patterns and in [5] for non-homogeneous linear patterns. Here we prove the result for ideals of numerical semigroups, making use of Abel's partial summation formula (this proof is due to Christian Gottlieb).

Lemma 14. *If $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i + a_0$ is a linear pattern admitted by an ideal I of a numerical semigroup S (i.e. $p(s_1, \ldots, s_n) \in S$ for all non-increasing sequences $s_1, \ldots, s_n$ of elements from I), then*

- $\sum_{i=1}^{n'} a_i \geq 0$ *for all* $1 \leq n' \leq n$, *and*
- $\sum_{i=1}^{n} a_i s_i \geq \sum_{i=1}^{n} a_i s_n$ *for all non-increasing sequences* $s_1, \ldots, s_n \in I$.

Proof. Assume there is an n' with $1 \leq n' \leq n$ such that $\sum_{i=1}^{n'} a_i < 0$. If s is very large compared to t we obtain

$$p(\underbrace{s, \ldots, s}_{n'}, \underbrace{t, \ldots, t}_{n-n'}) = \left(\sum_{i=1}^{n'} a_i \right) s + \left(\sum_{i=n'+1}^{n} a_i \right) t < 0,$$

implying that p cannot be a pattern admitted by I.

Now, let $A_j = \sum_{i=1}^{j} a_i$. By Abel's formula for summation by parts [1], we have $\sum_{i=1}^{n} a_i(s_i - s_n) = A_n(s_n - s_n) - \sum_{i=1}^{n-1} A_i(s_{i+1} - s_i) = -\sum_{i=1}^{n-1} A_i(s_{i+1} - s_i) \geq 0$. $\square$

Proposition 15. *A linear endopattern of a numerical semigroup S is simply a linear pattern defined by a polynomial $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_n + a_0$ admitted by S. Therefore*

- $\sum_{i=1}^{n'} a_i \geq 0$ *for all* $1 \leq n' \leq n$, *and*
- $a_0 \in S$.

Proof. Let p be a linear endopattern of S, then p is a pattern defined by a linear multivariate polynomial $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i + a_0$ such that evaluated on any non-increasing sequence of elements of S, the result is in S. In particular, $p(0, \ldots, 0) = a_0 \in S$. For the rest of the statement, apply Lemma 14. $\square$

Proposition 16. *A linear surjective endopattern p of a numerical semigroup S is necessarily homogeneous. If p is a premonic homogeneous endopattern of S, then p is always surjective.*

Proof. A surjective endopattern p of S is an endopattern of S, therefore, by Lemma 15, p is a linear pattern defined by a polynomial of the form $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i + a_0$ with $a_0 \in S$ and $\sum_{i=1}^{n'} a_i \geq 0$ for all $1 \leq n' \leq n$. But if $a_0 > 0$ then this gives $p(s_1, \ldots, s_n) > 0$ for all non-increasing sequences of S so that $p(S) \subsetneq S$. Consequently, if $p(S) = S$, then p is defined by a homogeneous linear pattern. Finally, if p is premonic then, by Lemma 5, p is surjective. □

The next result gives a necessary condition for when a polynomial defines a linear pattern admitted by a proper ideal of a numerical semigroup. When the ideal is a maximal ideal then this result strengthens the necessary condition given in [5].

Lemma 17. *If S is a numerical semigroup and $p = \sum_{i=1}^{n} a_i X_i + a_0$ is a linear pattern admitted by a proper ideal I of S, then*

- $\sum_{i=1}^{n'} a_i \geq 0$ *for all* $1 \leq n' \leq n$ *and, moreover,*
- $\sum_{i=1}^{n} a_i \geq \max(0, -a_0/\mu(I))$*, where* $\mu(I) = \min(I)$*.*

Proof. For the first part, apply Lemma 14. The second part is an improvement of $\sum_{i=1}^{n} a_i \geq 0$ which is relevant only when $a_0 < 0$. There are no linear patterns admitted by S with $a_0 < 0$ (see Proposition 15), but there may be linear patterns admitted by I with that property. Therefore assume that $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i + a_0$ is a linear pattern admitted by I with $a_0 < 0$ and $\sum_{i=1}^{n} a_i < \max(0, -a_0/\mu(I)) = -a_0/\mu(I)$, then $p(\mu(I), \ldots, \mu(I))) = \sum_{i=1}^{n} a_i \mu(I) + a_0 < (-a_0/\mu(I)) \cdot \mu(I) + a_0 = 0$ so that $p(\mu(I), \ldots, \mu(I)) \notin S$. But then p cannot be a pattern of I and we have a contradiction. □

We now use Lemma 17 to give necessary conditions for when a pattern is an endopattern of a proper ideal of a numerical semigroup.

Proposition 18. *A linear endopattern of a proper ideal I of a numerical semigroup S is a linear pattern $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i + a_0$ admitted by I, and so (by Lemma 17) p necessarily satisfies*

- $\sum_{i=1}^{n'} a_i \geq 0$ *for all* $1 \leq n' < n$*,*
- $\sum_{i=1}^{n} a_i \geq \max(0, -a_0/\mu(I))$*,*

and additionally

- $a_0 > 0$ *or* $\sum_{i=1}^{n} a_i > \max(0, -a_0/\mu(I))$ *(or both),*

where $\mu(I) = \min(I)$*.*

Proof. The first part of this result is Lemma 17. For the second part, assume that $a_0 \leq 0$ and $\sum_{i=1}^n a_i = \max(0, -a_0/\mu(I))$. Then $\sum_{i=1}^n a_i\mu(I) + a_0 = \max(0, -a_0/\mu(I))\mu(I) + a_0 = 0$, so that $p(I) \not\subseteq I$. $\square$

Proposition 19. *A linear pattern $p(X_1, \ldots, X_n) = \sum_{i=1}^n a_iX_i + a_0$ admitted by the maximal ideal $M(S)$ of a numerical semigroup S is an endopattern of $M(S)$ if and only if $a_0 > 0$ or $\sum_{i=1}^n a_i > \max(0, -a_0/m(S))$ (or both).*

Proof. By Proposition 18, if p is an endopattern, then $a_0 > 0$ or $\sum_{i=1}^n a_i > \max(0, -a_0/m(S))$ (or both).

Now assume that $p(X_1, \ldots, X_n) = \sum_{i=1}^n a_iX_i + a_0$ is a linear pattern admitted by $M(S)$. Then by Lemma 14, $\sum_{i=1}^n a_is_i \geq \sum_{i=1}^n a_is_n \geq 0$ for all non-increasing sequences $s_1, \ldots, s_n \in M(S)$. Therefore, if $a_0 > 0$, then $p(s_1, \ldots, s_n) = \sum_{i=1}^n a_is_i + a_0 \geq a_0 > 0$ for all non-increasing sequences $s_1, \ldots, s_n \in M(S)$, so that $p(M(S)) \subseteq M(S)$ and p is an endopattern of $M(S)$. Also, if $\sum_{i=1}^n a_i > \max(0, -a_0/m(S))$, then $p(s_1, \ldots, s_n) = \sum_{i=1}^n a_is_i + a_0 \geq (\sum_{i=1}^n a_i)m(S) + a_0 > \max(0, -a_0/m(S))m(S) + a_0 \geq 0$ for all non-increasing sequences $s_1, \ldots, s_n \in M(S)$, so that $p(M(S)) \subseteq M(S)$ and p is an endopattern of $M(S)$. $\square$

Proposition 20. *Any linear surjective endopattern of a proper ideal I of a semigroup S is necessarily of the form $p(X_1, \ldots, X_n) = \sum_{i=1}^n a_iX_i + a_0$ satisfying $a_0 = -(\sum_{i=1}^n a_i - 1)\mu(I)$ where $\mu(I)$ is the smallest element of I. Also, if p is a premonic endopattern of I, such that $a_0 = -(\sum_{i=1}^n a_i - 1)\mu(I)$, then p is surjective.*

Proof. Denote by $\mu(I)$ the smallest element of I. If $p = \sum_{i=1}^n a_iX_i + a_0$ is a linear surjective endopattern of I, then by Lemma 14 $\sum_{i=1}^n a_is_i \geq \sum_{i=1}^n a_is_n \geq \sum_{i=1}^n a_i\mu(I)$ for all non-increasing sequences of I. Since p is surjective, $\mu(I)$ is in $p(I)$, forcing $\sum_{i=1}^n a_i\mu(I) + a_0 = \mu(I)$ so that $a_0 = -(\sum_{i=1}^n a_i - 1)\mu(I)$.

Now, if p is premonic, then $\sum_{i=1}^j a_i = 1$ for some $j \leq n$ so that if $a_0 = -(\sum_{i=1}^n a_i - 1)\mu$, then $a_0 = -\sum_{i=j+1}^n a_i\mu$ and so $p(s, \ldots, s, \mu, \ldots, \mu) = \sum_{i=1}^j a_is + \sum_{i=j+1}^n a_i\mu + a_0 = s$ for all $s \in I$, implying that $p(I) = I$. $\square$

The linear patterns considered in the literature before this article are either homogeneous patterns admitted by S or non-homogeneous patterns admitted by $M(S)$. They all have the numerical semigroup S as codomain. The next result shows that almost all these patterns are also endopatterns of $M(S)$.

Corollary 21. *Let S be a numerical semigroup and $M(S)$ its maximal ideal.*

- *If $p(X_1, \ldots, X_n) = \sum_{i=1}^n a_iX_i$ is a homogeneous linear pattern admitted by S which is not an endopattern of $M(S)$, then $\sum_{i=1}^n a_i = 0$.*

- *If $p(X_1, \ldots, X_n) = \sum_{i=1}^n a_iX_i + a_0$ is a non-homogeneous linear pattern admitted by $M(S)$ which is not an endopattern of $M(S)$, then $a_0 \leq 0$ and $\sum_{i=1}^n a_i = -a_0/m(S)$.*

Proof.
- If p is a homogeneous linear pattern admitted by S, then p is also admitted by $M(S)$. By Lemma 17 and Proposition 19, if p is not an endopattern then $\sum_{i=1}^n a_i = 0$.
- If p is a non-homogenous linear pattern admitted by $M(S)$, then by Lemma 17, Proposition 18 and Proposition 19, if p is not an endopattern then $a_0 \leq 0$ and $\sum_{i=1}^n a_i = -a_0/m(S)$.

□

Examples of patterns admitted by a maximal ideal $M(S)$ of a semigroup S that are not endopatterns of $M(S)$ can be found in the two non-homogeneous patterns in Weierstrass semigroups mentioned in the introduction.

Corollary 21 shows that many of the important patterns previously considered in the literature are endopatterns of $M(S)$. For example, this is true for the Arf pattern, the subtraction patterns and the patterns of the form $X + a$ with a pseudo-Frobenius. They all belong to the important class of monic linear patterns.

Lemma 22. *Let S be a numerical semigroup. If $S \neq \mathbb{Z}_+$ and $a_0 \notin M(S)$, then there are no monic linear patterns $p(X_1, \ldots, X_n) = \sum_{i=1}^n a_i X_i + a_0$ admitted by S or by its maximal ideal $M(S)$ with $\sum_{i=1}^n a_i = \max(0, -a_0/m(S))$.*

Proof. Let $m = m(S)$ and let p be a monic linear pattern with $\sum_{i=1}^n a_i = \max(0, -a_0/m)$.

If $a_0 \leq 0$, then $\max(0, -a_0/m) = -a_0/m$. Let s be the smallest element of $M(S)$ that is not a multiple of m, then $p(s, m, \ldots, m) = s + \sum_{i=2}^n a_i m + a_0 = (s - m) + p(m, \ldots, m) = s - m + \max(0, -a_0/m)m + a_0 = s - m \in S$. Now $s - m < s$ and $s - m$ is not a multiple of m, but s is the smallest element in $M(S)$ that is not a multiple of $m > 1$, so there is a contradiction. So there are no monic linear patterns admitted by S (or $M(S)$) with $\sum_{i=1}^n a_i = \max(0, -a_0/m)$ and $a_0 \leq 0$.

If $a_0 > 0$, then $\max(0, -a_0/m) = 0$, implying that for all $s \in M(S)$ we have $p(s, \ldots, s) = \sum_{i=1}^n a_i s + a_0 = a_0 \in S$, and since $a_0 > 0$, we have $a_0 \in M(S)$. Therefore, there are no monic linear patterns admitted by S (or $M(S)$) with $\sum_{i=1}^n a_i = \max(0, -a_0/m)$, $a_0 > 0$ and $a_0 \notin M(S)$. □

Corollary 23. *If $S \neq \mathbb{Z}_+$, then any monic linear pattern admitted by $M(S)$ is an endopattern of $M(S)$.*

Proof. If p is a monic linear pattern admitted by $M(S)$, then, by Lemma 17 and Lemma 22, $\sum_{i=1}^n a_i > \max(0, -a_0/m(S))$. Therefore, by Proposition 19, p is an endopattern of $M(S)$. □

5 Closures of ideals with respect to linear patterns

A pattern is admissible if it is admitted by some numerical semigroup. In [4] the closure of a numerical semigroup S with respect to an admissible homogeneous

pattern p was defined as the smallest numerical semigroup that admits p and contains S. Here this definition is generalised to non-homogeneous patterns and to ideals of numerical semigroups.

Definition 24. *Given an ideal I of a numerical semigroup S and an admissible pattern p not necessarily admitted by I, define the closure of I with respect to p as the smallest ideal $\tilde{I}$ of some numerical semigroup $\tilde{S}$ that admits p and contains I.*

It is not required that I is an ideal of $\tilde{S}$, nor is it required that $\tilde{I}$ is an ideal of S. However, by definition, it is always true that $I \subseteq \tilde{I} \subseteq \tilde{S}$.

Note that if I is not contained in any ideal of a numerical semigroup that admits p, then the closure of I with respect to p will fail to exist. This is not a problem for homogeneous linear patterns since a homogeneous pattern p is admissible if and only if p is admitted by $\mathbb{Z}_+$ [4]. Therefore, if p is admissible then there is always an ideal of a numerical semigroup that admits p and contains I.

An ordinary numerical semigroup is a numerical semigroup of the form $\{0, m, \rightarrow\}$. From [5], Theorem 3.7, we know that if p is an admissible non-homogeneous linear pattern then there is an ordinary numerical semigroup that admits p. If μ is the smallest integer such that $\{0, \mu, \rightarrow\}$ admits p, then we say that p is μ-admissible.

Lemma 25. *The closure of an ideal I of a numerical semigroup with respect to an admissible linear pattern p is well-defined if p is μ-admissible for $\mu \leq \min(I)$.*

Proof. If p is μ-admissible and $\min(I) \geq \mu$, then $I \subseteq \{0, \mu, \rightarrow\}$ so there is an ideal of a numerical semigroup that contains I and admits p, implying that the closure of I with respect to p is well-defined. □

Note that the closure of I with respect to p can be well-defined although p is not μ-admissible for $\mu \leq \min(I)$. The smallest m such that $\{0, m, \rightarrow\}$ admits the linear pattern $p(X_1) = X_1 + X_2 - 3$ is $m = 3$, so p is 3-admissible. However, the ideal $\{2, 3, \rightarrow\}$ of the numerical semigroup $\mathbb{Z}_+$ also admits p. Therefore the closure of I with respect to p is well-defined for any ideal I with $\min(I) \geq 2$.

It was proved in [4] that if p is a premonic homogeneous linear pattern, then the closure of S with respect to p can be calculated as

$$\underbrace{p(p(\cdots(p}_{k}(S))\cdots)),$$

denoted as $p^k(S)$, for some k large enough. The next result generalises this to premonic non-homogeneous patterns and proper ideals of numerical semigroups.

Theorem 26. *If I is an ideal of a numerical semigroup and $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_i X_i + a_0$ is a premonic linear pattern satisfying $a_0 = -(\sum_{i=1}^{n} a_i - 1)\mu$ with $\mu = \min(I)$, then $I \subseteq p(I)$ and the chain $I_0 = I \subseteq I_1 = p(I_0) \subseteq I_2 = p(I_2) \subseteq \cdots$ stabilizes. The ideal $I_k = p^k(I)$ for k such that $p^{k+1}(I) = p^k(I)$ is the closure of I with respect to p.*

Proof. As in the proof of Proposition 20, since p is premonic, $\sum_{i=1}^{j} a_i = 1$ for some $j \leq n$, so that $a_0 = -(\sum_{i=1}^{n} a_i - 1)\mu = -\sum_{i=j+1}^{n} a_i\mu$. Therefore $p(s, \ldots, s, \mu, \ldots, \mu) = \sum_{i=1}^{j} a_i s + \sum_{i=j+1}^{n} a_i\mu + a_0 = s$ for all $s \in I$, implying that $I \subseteq p(I)$. Note also that since p is admissible, by Lemma 14, $p(s_1, \ldots, s_n) = \sum_{i=1}^{n} a_i s_i + a_0 \geq \sum_{i=1}^{n} a_i\mu + a_0 = \mu$ for all non-increasing sequences $s_1, \ldots, s_n \in \{\mu, \rightarrow\}$, implying that $p^k(I) \subseteq \{\mu, \rightarrow\}$ for all $k \geq 1$. The ideal I has finite complement in $\mathbb{Z}_+$, implying that the chain

$$I_0 = I \subseteq I_1 = p(I_0) \subseteq I_2 = p(I_2) \subseteq \cdots$$

stabilizes. Clearly if $p^{k+1}(I) = p^k(I)$ then $p^k(I)$ is an ideal of S that admits p and contains I. Finally, if J is the closure of I with respect to p, then J must contain $p^i(I)$ for all $i \geq 1$, so that J contains $p^k(I)$. Therefore $p^k(I)$ is the smallest ideal of S that admits p and contains I, so $p^k(I)$ is the closure of I with respect to p. □

Note that the conditions on the pattern in Theorem 26 are the same as the sufficient conditions for surjective endopatterns in Proposition 20.

6 Giving structure to the set of patterns admitted by a numerical semigroup

A numerical semigroup admits in general many patterns. These patterns can be combined in several ways.

Lemma 27. *Let I be an ideal of a numerical semigroup S and suppose that p and q are two patterns admitted by I. Then $p + q$ and rp are also patterns admitted by I for any polynomial r with coefficients in $\mathbb{Z}$ such that $r(I) \geq 0$ when evaluated on any non-increasing sequence of elements from I.*

Proof. For all $s_1, \ldots, s_n \in I$ we have $p(s_1, \ldots, s_n) + q(s_1, \ldots, s_n) = a + b$ for some $a, b \in S$, so that $a + b \in S$, implying that $p + q$ is a pattern admitted by I. Also, $r(s_1, \ldots, s_n)p(s_1, \ldots, s_n) = ab$ for some $a \geq 0$ and $b \in S$. Since ab is the result of adding b to itself a times we have that $ab \in S$, implying that rp is a pattern admitted by I. □

It can be argued that since a numerical semigroup is an additive structure, the linear patterns are the most important patterns. Note that linearity is necessary for the pattern to preserve the additivity of the numerical semigroup.

Denote by $\mathcal{P}_n^d(I)$ the set of patterns of length at most n and degree at most d that are admitted by the ideal I of a numerical semigroup S. Then $\mathcal{P}_n^1(I)$ is the set of linear patterns of length at most n admitted by I. Lemma 27 gives algebraic structure to $\mathcal{P}_n^d(I)$.

Lemma 28. *Let I be an ideal of a numerical semigroup S, $n \geq 0$ and $d \geq 0$. Then $\mathcal{P}_n^d(I)$ is a semigroup with zero, a monoid.*

Proof. By Lemma 27, if p and q are patterns admitted by I, then $p+q$ is a pattern admitted by I. Also, if p and q are of length at most n and degree at most d, then $p+q$ is a pattern of length at most n and degree at most d. Therefore $\mathcal{P}_n^d(I)$ is a semigroup with respect to addition. The zero pattern is admitted by any ideal of any numerical semigroup and has length at most n and degree at most d for any $n \geq 0$ and $d \geq 0$, so $0 \in \mathcal{P}_n^d(I)$. $\square$

The set $\mathcal{P}_n^d(I)$ is not preserved by polynomial multiplication, but, if so is desired, this problem can be overcome by instead considering patterns of arbitrary degree. Denote by $\mathcal{P}_n(I)$ the set of patterns of length at most n that are admitted by I.

A semiring is a set X together with two binary operations called addition and multiplication such that X is a semigroup with both addition and multiplication, and multiplication distributes over addition. In general X is not required to have neither zero nor unit element.

Lemma 29. *Let I be an ideal of a numerical semigroup S. Then $\mathcal{P}_n(I)$ is a semiring with zero element. There is a unit element if and only if $I = \mathbb{Z}_+$.*

Proof. By Lemma 27, if p and q are patterns admitted by I, then $p+q$ and pq are patterns admitted by I, so $\mathcal{P}_n(I)$ is a semigroup with respect to both addition and multiplication. Also, clearly multiplication distributes over addition. Note that the pattern $p(X_1, \ldots, X_n) = 0$ always is a pattern admitted by S. The semiring $\mathcal{P}_n(I)$ has a unit if and only if $1 \in \mathcal{P}_n(I)$, which happens if and only if $I = S = \mathbb{Z}_+$. $\square$

If Σ is a (commutative) semiring with unit, then a semiring X is a semialgebra over Σ if there is a composition $(\sigma, x) = \sigma x$ from $\Sigma \times X$ to X such that $(X, +)$ is a (left) Σ-semimodule with $(\sigma, x) = \sigma x$ and for $\sigma \in \Sigma$ and $x, y \in X$, $\sigma(xy) = (\sigma x)y = x(\sigma y)$. The semigroup $(X, +)$ is a (left) Σ-semimodule if $\sigma(x+y) = \sigma x + \sigma y$, $(\sigma + \rho)x = \sigma x + \rho y$, $(\sigma\rho)x = \sigma(\rho x)$ and $1 \cdot x = x$ for all $\sigma, \rho \in \Sigma$ and for all $x, y \in X$.

Lemma 30. *Let I be an ideal of a numerical semigroup S and consider the set of polynomials $R(I) = \{r \in \mathbb{Z}[X_1, \ldots, X_n] : r(s_1, \ldots, s_n) \geq 0 \ \forall s_1 \geq \cdots \geq s_n \in I\}$. Then $R(I)$ is a semiring (with zero and unit elements) and $\mathcal{P}_n(I)$ is an $R(I)$-semialgebra.*

Proof. Following the proof of Lemma 29, we see that $R(I)$ is a semiring with zero and unit elements. By Lemma 27 we have that $rp \in \mathcal{P}_n(I)$. Also, $r(pq) = (rp)q = p(rq)$ for all $r \in R(I)$ and for all $p, q \in \mathcal{P}_n(I)$. Now let r and s be elements in $R(I)$ and p and q be elements in $\mathcal{P}_n(I)$. Then it can easily be checked that $r(p+q) = rp + rq$, $(r+s)p = rp + sp$, $(rs)p = r(sp)$ and $1 \cdot p = p$, implying that $\mathcal{P}_n(I)$ is an $R(I)$-semimodule. By Lemma 29, $\mathcal{P}_n(I)$ is also a semiring and consequently an $R(I)$-semialgebra. $\square$

7 Linear patterns and a generalisation of pseudo-Frobenius

Let J be an ideal of a numerical semigroup S and let p be an endopattern of J. We will now study sufficient conditions on a_0 for when p induces the pattern $p + a_0$ on the ideals of S contained in J. We are also interested in when this implies that $p + a_0$ is an endopattern of J. Finally we will also study sufficient conditions for when the endopatterns $p_1, \ldots, p_n$ induce the pattern $p_1 + \cdots + p_n + a_0$ on the ideals contained in J.

Lemma 31. *If p is an endopattern of S and $a_0 \in S$, then p induces the pattern $p + a_0$ on any ideal J of S. Additionally, $p + a_0$ is an endopattern of S (but not necessarily of other ideals of S).*

Proof. If $p(s_1, \ldots, s_n) \in S$ and $a_0 \in S$, then $p(s_1, \ldots, s_n) + a_0 \in S$, so $p + a_0$ is admitted by S and by any ideal J of S. Endopatterns of S are simply patterns admitted by S. □

In other words, an endopattern p of a numerical semigroup S induces the pattern $p + a_0$ on an ideal J under the condition that (i) $a_0 \in S$ and (ii) the ideal J is an ideal of S. It is (of course) not true that if p is an endopattern of S and $a_0 \in S$ then p induces $p + a_0$ on any ideal of any numerical semigroup.

Lemma 32. *If p is an endopattern of $M(S)$ and $a_0 \in PF(S)$, then p induces $p + a_0$ on any ideal $J \subseteq M(S)$. Additionally, if $S \neq \mathbb{Z}_+$, then $p + a_0$ is an endopattern of $M(S)$ (but not necessarily of other ideals contained in $M(S)$.*

Proof. By definition of pseudo-Frobenius the monic linear pattern $f(X) = X + a_0$ is admitted by $M(S)$, implying that $f(p) = p + a_0$ is admitted by $M(S)$ and by any ideal of S contained in $M(S)$. By Corollary 23, since $S \neq \mathbb{Z}_+$, f is an endopattern of $M(S)$, implying that $f(p) = p + a_0$ is an endopattern of $M(S)$. □

Again, this means that an endopattern p of a maximal ideal $M(S)$ of a numerical semigroup S induces the pattern $p + a_0$ on an ideal J under the condition that (i) $a_0 \in PF(S)$ and (ii) the ideal $J \subseteq M(S)$.

Consider for example the numerical semigroup S generated by 2 and 5. There are no other pseudo-Frobenius than the Frobenius element, so $PF(S) = \{3\}$. Any numerical semigroup admits the trivial pattern defined by $p(X_1) = X_1$, which is always an endopattern of the maximal ideal, and consequently the pattern $X_1 + 3$ is admitted by any ideal of S contained in $M(S)$. Also, $X_1 + 3$ is an endopattern of $M(S)$.

Note that if $a_0 \notin S \cup PF(S)$, then an endopattern p of $M(S)$ does not necessarily induce the pattern $p + a_0$ on $M(S)$. For example, the numerical semigroup $S = \langle 2, 7\rangle = \{0, 2, 4, 6, \rightarrow\}$ has $PF(S) = \{5\}$. We have that $M(S)$ admits the Arf endopattern $X_1 + X_2 - X_3$ and the non-homogeneous endopattern $X_1 + X_2 - X_3 + 5$, but $M(S)$ does not admit $X_1 + X_2 - X_3 + 3$. However, note that $1 \notin S \cup PF(S)$ but $X_1 + X_2 + X_3 + 1$ is an endopattern of $M(S)$.

We have seen that the pseudo-Frobenius $PF(S)$ of a numerical semigroup S are related to the linear endopatterns $X_1 + a_0$ of $M(S)$, with $a_0 \in PF(S)$. By replacing the variable X_1 by an endopattern p of $M(S)$ this resulted in a statement on for which $a_0 \in \mathbb{Z}$ p induces the pattern $p + a_0$. We will now generalise this idea in more than one direction, to sums of several patterns and to any ideal of a numerical semigroups.

Definition 33. *Let I and J be two ideals of the same numerical semigroup S. For $d \geq 1$, define the set $PF^d(I, J) = (I - dJ) \setminus (I - (d-1)J)$ and call it* the set of elements at distance d from I with respect to J.

The elements at distance zero from S with respect to any ideal J of S, $PF^0(S, J)$, can be defined to be the elements in S, if so desired. The elements at distance n from S with respect to S, $PF^n(S, S)$, is the empty set when $n \geq 1$, reflecting the fact that the linear pattern $X_1 + \cdots + X_d + a$ is admitted by S if and only if $a \in S$ for all $d \geq 0$. The elements at distance one from S with respect to $M(S)$, $PF^1(S, M(S)$, is the set of pseudo-Frobenius of S, and if $S \neq \mathbb{Z}_+$, then, by Corollary 23, we have $PF^1(M(S), M(S)) = PF^1(S, M(S)) = PF(S)$.

Note that $PF^d(S, J)$ are the elements $a \in \mathbb{Z}$ such that for any collection of d (but not for any collection of $d-1$) endopatterns $q_1, \ldots, q_d$ of J, the pattern $q_1 + \cdots + q_d + a$ is also a pattern admitted by J, and therefore by any ideal contained in J. In general we have the following.

Lemma 34. *Let I and J be two ideals of the same numerical semigroup S, let $p_1, \ldots, p_d$ be endopatterns of J and let $a_0 \in PF^d(I, J)$. Then the pattern $q = p_1 + \cdots + p_n + a_0$ is admitted by any ideal $K \subseteq J$ and its image satisfies $q(K) \subseteq I$. In particular, if $I \subseteq K$, then q is an endopattern of K.*

Proof. It is clear from the definition of $PF^d(I, J)$ that the pattern $X_1 + \cdots + X_d + a_0$ is admitted by any ideal K contained in J, and that its image is contained in I. The result follows from substituting $X_1, \ldots, X_d$ with the endopatterns $p_1, \ldots, p_d$ of J. □

The Lipman semigroup of S with respect to a proper ideal J is $L(S, J) = \bigcup_{h \geq 1}(hJ - hJ)$ [8, 2]. The semigroup $L(S) := L(S, M(S))$ is called the Lipman semigroup of S. There exists a $h_0 \geq 1$ such that $L(S, J) = (hJ - hJ)$ for each $h \geq h_0$, and, for each $h \geq h_0$, $(h+1)J = hJ + \mu(J)$ where $\mu(J) = \min(J)$ [2].

Proposition 35. *When S is of maximal embedding dimension, then*

$$PF^2(S, M(S)) = E(S) - 2m(S).$$

Proof. Define $D(d, M(S)) = \{z \in \mathbb{Z} : z + dM(S) \subseteq S, z + dM(S) \not\subseteq dM(S)\}$, that is, $D(d, M(S)) = \{z \in \mathbb{Z} : z + dM(S) \subseteq S, (z + dM(S)) \cap (S \setminus dM(S)) \neq \emptyset\}$. Note that $S \setminus M(S) = \{0\}$, so that $D(1, M(S)) = \{z \in \mathbb{Z} : z + M(S) \subseteq S, 0 \in z + M(S)\}$. Also, note that $S \setminus 2M(S) = E(S)$ is the set of minimal generators of S. By definition, if d is such that $L(S) = (dM(S) - dM(S))$ then $(S - dM(S)) = L(S) \cup D(d, M(S))$.

When S is of maximal embedding dimension, then the Lipman semigroup of S is $L(S) = (hM(S) - hM(S))$ for all $h \geq 1$, so that

$$\begin{aligned} PF^2(S, M(S)) \quad &= (S - 2M(S)) \setminus (S - M(S)) \\ &= D(2, M(S)) \setminus D(1, M(S)) \\ &= \{z \in \mathbb{Z} : z + 2M(S) \subseteq S, z + 2M(S) \nsubseteq 2M(S), z + M(S) \nsubseteq S\} \\ &= E(S) - 2m(S). \end{aligned}$$

□

Compare this with the fact that the pattern $X_1 + X_2 - m(S)$ is admitted by a numerical semigroup if and only if S is of maximal embedding dimension, and note that the smallest element in the set $E(S) - 2m(S)$ is $-m(S)$.

Theorem 36. *The cardinality of $PF^d(I, J)$ converges to $\mu(J) = \min(J)$. When J is a proper ideal then the convergence follows the convergence of the Lipman semigroup of S with respect to I.*

Proof. First note that $\min(S) = 0$ and $PF^d(S, J) = \emptyset$ for $d \geq 1$ and for any ideal J of S.

If J is a proper ideal of S, then consider the Lipman semigroup with respect to J, $L(S, J) = \bigcup_{h \geq 1}(hJ - hJ)$. There is a smallest $h_0 \geq 1$ such that $L(S, J) = hJ - hJ$ whenever $h \geq h_0$, in which case we also have $(h+1)J = hJ + \mu(J)$ (see Proposition I.2.1 in [2]). Therefore, if $d \geq h_0$ then $z+(d+1)J = z+\mu(J)+dJ$ for $z \in \mathbb{Z}$ implying that $z+(d+1)J \subseteq I$ and $z+dJ \nsubseteq I$ if and only if $(z+\mu(J))+dJ \subseteq I$ and $(z+\mu(J))+(d-1)J \nsubseteq I$. Consequently $(I-(d+1)J) = (I-dJ)-\mu(J)$ so that $PF^{d+1}(I, J) = (I-(d+1)J) \setminus (I-dJ) = ((I-dJ)-\mu(J)) \setminus (I-dJ)$, which has cardinality $\mu(J)$. □

This is not the only way to generalise the notion of pseudo-Frobenius. Let $S = \{0 = s_0, s_1, \ldots, s_n, \rightarrow\}$ be a numerical semigroup with conductor s_n. For $1 \leq i \leq n$, consider the ideal $S_i = \{s \in S : s \geq s_i\}$, let $S(i) = S_i^* = (S - S_i)$ be its dual relative ideal, and let $T_i(S) = S(i) \setminus S(i-1)$. The *type sequence* of a numerical semigroup S is the finite sequence $(|T_i(S)| : 1 \leq i \leq n)$ [2]. Since $T_1 = PF$ and $|PF|$ is the type of S, this is a generalisation of pseudo-Frobenius, which is different from the one in this article.

Next we will give examples of how the sets $PF^d(I, J)$ can be used to understand small semigroups of linear patterns better.

Example 37. *Let S be a numerical semigroup. Then, by definition, $M(S)$ admits the pattern $p_d(X_1, \ldots, X_n) = \sum_{i=1}^d X_i + a_0$ for any $a_0 \in PF^d(S, M(S))$. Clearly the pattern p_d induces the pattern $q_d(X_1) = dX_1 + a_0$. Therefore $\{p(X_1) = a_1X_1 + a_0 \in \mathbb{Z}[X_1] : a_1 \geq 0, a_0 \in S \cup \bigcup_{i=1}^{a_1} PF^i(S, M(S))\} \subseteq \mathcal{P}_1^1(M(S))$.*

Example 38. *Let S be an ordinary numerical semigroup, so that $z \in S$ for all $z \in \mathbb{Z}$ such that $z \geq m(S)$. Then, if $q_n(X_1) = nX_1 + a_0$ is a pattern of S, so that $nm(S) + a_0 \in M(S)$, we have that $p_n(s_1, \ldots, s_n) = \sum_{i=1}^n s_i + a_0 \geq$*

$\sum_{i=1}^{n} m(S) + a_0 = nm(S) + a_0$ *so that* $p_n(s_1, \ldots, s_n) \in M(S)$*, implying that* $p_n(X_1, \ldots, X_n) = \sum_{i=1}^{n} X_i + a_0$ *is also a pattern of* S*, and so* p_n *and* q_n *are equivalent. Therefore* $\mathcal{P}_1^1(S) = \{p(X_1) = a_1X_1 + a_0 \in \mathbb{Z}[X_1] : a_1 \geq 0, a_0 \in S \cup \bigcup_{i=1}^{a_1} PF^i(S, M(S))\}$.

Note that if S is not ordinary, then in general it is not true that $\mathcal{P}_1^1(S) = \{p(X_1) = a_1X_1 + a_0 \in \mathbb{Z}[X_1] : a_1 \geq 0, a_0 \in S \cup \bigcup_{i=1}^{a_1} PF^i(S, M(S))\}$. For example, if $S = \langle 3, 5\rangle$, then $2X_1 - 1$ is a pattern, but $-1 \in PF^3(S, M(S))$, in particular $-1 \notin PF^i(S, M(S))$ for $i \leq 2$.

8 Numerical semigroups as the image of other numerical semigroups under linear patterns

We saw in Corollary 4 that if $p(X_1, \ldots, X_n) = \sum_{i=1}^{n} a_iX_i$ is a homogeneous pattern admitted by the numerical semigroup S then $p(S)$ is a numerical semigroup if and only if $\gcd(a_1, \ldots, a_n) = 1$. However, neither Theorem 3 nor Corollary 4 say anything about the numerical semigroup $p(S)$. Clearly, any numerical semigroup is the image of some numerical semigroup under some pattern. Indeed, any numerical semigroup is the image of itself under the pattern $p(X) = X$.

Lemma 39. *Any numerical semigroup* $S = \langle a_1, \ldots, a_e\rangle$ *is the image of* $\mathbb{Z}_+$ *under the homogeneous pattern* $p(X_1, \ldots, X_e) = a_1X_1 + \sum_{i=2}^{e}(a_i - a_{i-1})X_i$.

Proof. Let $S = \langle a_1, \ldots, a_e\rangle$ be a numerical semigroup, with $a_1 \geq \cdots \geq a_e$ a (not necessarily minimal) set of generators of S. Let $p(X_1, \ldots, X_e) = a_1X_1 + \sum_{i=2}^{e}(a_i - a_{i-1})X_i$. Then for any non-increasing sequence $s_1, \ldots, s_e \in \mathbb{Z}_+$ we have $p(s_1, \ldots, s_e) = a_1s_1 + \sum_{i=2}^{e}(a_i - a_{i-1})s_i = \sum_{i=1}^{e-1} a_i(s_i - s_{i+1}) + a_es_e$ and since $s_i \geq s_{i+1}$ for all $i \in 1, \ldots, e-1$ we have $s_i - s_{i+1} \geq 0$ so that $p(s_1, \ldots, s_e) \geq 0$ and therefore p is a homogeneous pattern admitted by $\mathbb{Z}_+$. Moreover, since $p(s_1, \ldots, s_n)$ is of the form $\sum_{i=1}^{e} a_in_i$ with $n_i \geq 0$ we have that $p(s_1, \ldots, s_e) \in \langle a_1, \ldots, a_e\rangle = S$ so that $p(\mathbb{Z}_+) \subseteq S$. Now, for each generator a_j of S, the non-increasing sequence

$$s_1, \ldots, s_e = \overbrace{1, \ldots, 1}^{j}, \overbrace{0, \ldots, 0}^{e-j}$$

gives $p(s_1, \ldots, s_n) = \sum_{i=1}^{j-1} a_i(1-1) + a_j(1-0) + \sum_{i=e-j}^{e-1}(0-0) + a_e0 = a_j$, so that $a_j \in p(\mathbb{Z}_+)$. Finally, since p is linear, $p(x_1, \ldots, x_e) + p(y_1, \ldots, y_e) = p(x_1 + y_1, \ldots, x_e + y_e)$ for all sequences $x_1, \ldots, x_e$ and $y_1, \ldots, y_e$ in $\mathbb{Z}_+$. Note that if $x_1, \ldots, x_e$ and $y_1, \ldots, y_e$ are non-increasing sequences of $\mathbb{Z}_+$ then so is $x_1 + y_1, \ldots, x_e + y_e$. Therefore, for all $a, b \in p(\mathbb{Z}_+)$, also $a + b \in p(\mathbb{Z}_+)$. (Compare the proof of Lemma 4.) Consequently, we have $S = \langle a_1, \ldots, a_e\rangle \subseteq p(S)$, implying $S = p(\mathbb{Z}_+)$. $\square$

Note that if the numerical semigroup S is the image of a numerical semigroup $S' \supseteq S$ under a pattern p, then S admits p. Therefore it is possible to consider

the chain of numerical semigroups $S_0 = S \supseteq S_1 = p(S_0) \supseteq S_2 = p(S_1) \supseteq \cdots$. Observe that $p(p(S))$ is not the same as $p \circ (p, \ldots, p)(S)$ (see Section 9) and that this chain is not the same as the chain of numerical semigroups obtained in the closure of a numerical semigroup (see Definition 24 and [4]). Indeed, in the closure of a numerical semigroup S under a pattern p, the pattern is not necessarily admitted by the numerical semigroup, or, more precisely, it is only required that p is admissible (i.e. admitted by some numerical semigroup) and that $S \subseteq p(S)$. Then p is admitted by S if and only if S is the closure of S under p, in which case $p(S) = S$.

Now consider for a pattern p admitted by a numerical semigroup S the chain $S_0 = S \supseteq S_1 = p(S_0) \supseteq S_2 = p(S_1) \supseteq \cdots$. The chain either stabilizes to some numerical semigroup or it does not. If it stabilizes, then it does so at once, in which case p is a surjective endopattern of S. If it does not stabilize, then we want to explore relations between the consecutive numerical semigroups in the chain. The next result gives such a relation, under special conditions and when the length of the pattern is two.

The quotient of a numerical semigroup S by a positive integer d is the numerical semigroup $\frac{S}{d} = \{x \in \mathbb{Z}_+ : dx \in S\}$ [12].

Lemma 40. *Let S be a numerical semigroup and let $p(X_1, X_2) = a_1X_1 + a_2X_2$ be a linear homogeneous pattern in two variables (not necessarily admitted by S) such that $a_1 \in S$ and $\gcd(a_1, a_2) = 1$. Then $S = \frac{p(S)}{a_1+a_2}$.*

Proof. For all $s \in S$ we have $p(s, s) = (a_1 + a_2)s$, so that $S \subseteq \frac{p(S)}{a_1+a_2}$. Let $x \in \frac{p(S)}{a_1+a_2}$. Then there are $s_1, s_2 \in S$ such that $p(s_1, s_2) = a_1s_1 + a_2s_2 = (a_1 + a_2)x$, implying $a_1(s_1 - x) = a_2(x - s_2)$. By assumption $\gcd(a_1, a_2) = 1$, and so a_1 must divide $x - s_2$. Assume that $x < s_2$, then $a_1s_1 + a_2s_2 = (a_1 + a_2)x < (a_1 + a_2)s_2 < a_1s_1 + a_2s_2$, but that is impossible, and therefore $x \geq s_2$ and $x - s_2 \geq 0$. Now since a_1 divides $x - s_2$ and $a_1, s_1 \in S$, it follows that $x \in S$. Therefore $\frac{p(S)}{a_1+a_2} \subseteq S$. $\square$

It was proved in [11] that every numerical semigroup is one half of infinitely many symmetric numerical semigroups. This result was extended in [15] to numerical semigroups that are the quotient of infinitely many symmetric numerical semigroups by an arbitrarily integer $d \geq 2$. The much weaker result that every numerical semigroup is one divided by d of infinitely many numerical semigroups is easy to prove, just take $dS \cup \{ds + n : s \in S\}$ for distinct positive integers n with $\gcd(n, d) = 1$. However, we think that in light of Lemma 40, it is interesting to see that if S is a numerical semigroup, then the numerical semigroups $p(S)$ given by the linear homogeneous patterns of length two admitted by S of the form $p(X_1, X_2) = a_1X_1 + a_2X_2$, with $a_1 + a_2 = d$, $a_1 \in S$ and $\gcd(a_1, a_2) = 1$ so that $S = \frac{p(S)}{d}$, are all different. In other words, we let Lemma 40 imply that every numerical semigroup is the quotient of infinitely many numerical semigroups by an arbitrarily integer $d \geq 2$.

Corollary 41. *Let d be an integer satisfying $d \geq 2$. Any numerical semigroup S is the quotient from division by d of infinitely many numerical semigroups of*

the form $p(S)$ where p is a pattern of length two admitted by S. More precisely, we have that $S = \frac{p(S)}{d}$ for all $p(X_1, X_2) = a_1X_1 + a_2X_2$ such that $a_1 + a_2 = d$, $a_1 \in S$ and $\gcd(a_1 + a_2) = 1$.

Proof. By Lemma 40, any numerical semigroup S is the quotient from division by d of the numerical semigroup obtained as the image of S by any pattern of the form $p(X_1, X_2) = a_1X_1 + a_2X_2$ with $a_1 + a_2 = d$, $a_1 \in S$ and $\gcd(a_1 + a_2) = 1$. (Note that since $a_1 + a_2 \geq 0$ and $\gcd(a_1, a_2) = 1$, any pattern of a numerical semigroup of this form satisfies $d = a_1 + a_2 \geq 1$.) There is only a finite number of pairs (a_1, a_2) with $a_1, a_2 > 0$ and $a_1 + a_2 = d$, but there are infinitely many pairs (a_1, a_2) with $a_1 > 0$, $a_2 < 0$, $a_1 \in S$, $\gcd(a_1, a_2) = 1$ and $a_1 + a_2 = d$. Let α_1 be the smallest a_1 such that there is an $a_2 < 0$ with $a_1 + a_2 = d$ and let $\alpha_2 = d - \alpha_1$. Then $\alpha_1 = d + 1$ and $\alpha_2 = 1$. The other pairs (a_1, a_2) with $a_1 > 0$, $a_2 < 0$ and $a_1 + a_2 = d$ are obtained as $(a_1, a_2) = (\alpha_1 + k, \alpha_2 - k)$ with $k \in \mathbb{Z}_+$. Note that not all these pairs $(a_1, a_2) = (\alpha_1 + k, \alpha_2 - k)$ satisfy $\gcd(a_1, a_2) = 1$. More precisely, $\gcd(a_1, a_2) = 1$ if and only if $\gcd(a_1, d) = 1$. Indeed, any factor of a_1 divides $d = a_1 + a_2$ if and only if it divides a_2.

Let $q_k(X_1, X_2) = (\alpha_1 + k)X_1 + (\alpha_2 - k)X_2$. Clearly the set $D = \{ds : s \in S\} = \{q_k(s, s) : s \in S\} \subseteq q_k(S)$ for all $k \in \mathbb{Z}_+$. Therefore, if $q_k(S) \neq q_{k'}(S)$, then they differ in the elements outside D. The elements in $q_k(S) \setminus D$ are of the form $q_k(s_1, s_2)$ with $s_1 > s_2$, so that $s_1 - s_2 > 0$. Therefore, for any $k, k' \in \mathbb{Z}_+$ with $k' > k$ we have $q_{k'}(s_1, s_2) = (\alpha_1 + k')s_1 + (\alpha_2 - k')s_2 = \alpha_1 s_1 + \alpha_2 s_2 + k'(s_1 - s_2) > \alpha_1 s_1 + \alpha_2 s_2 + k(s_1 - s_2) = (\alpha_1 + k)s_1 + (\alpha_2 - k)s_2 = q_k(s_1, s_2)$.

Now assume that $\gcd(\alpha_1 + k, d) = 1$ (so that $\gcd(\alpha_1 + k, \alpha_2 - k) = 1$ and $q_k(S)$ is a numerical semigroup). Let $t_k = (\alpha_1 + k)s_1 + (\alpha_2 - k)s_2$ be the smallest element in $q_k(S)$ which is not of the form dn for $n \in \mathbb{Z}_+$, that is, the smallest element in $q_k(S)$ which is not divisible with $d = \alpha_1 + \alpha_2$. (Note that we proved in Lemma 40 that if $dn \in q_k(S)$, then $n \in S$ so that $dn = q_k(n, n)$.) Suppose that for some $k' > k$ we have $t_k \in q_{k'}(S)$. Then there are $s'_1, s'_2 \in S$ such that $t_k = q_{k'}(s'_1, s'_2) = (\alpha_1 + k')s'_1 + (\alpha_2 - k')s'_2 = (\alpha_1 + k)s'_1 + (\alpha_2 - k)s'_2 + (k' - k)(s'_1 - s'_2)$. But $(\alpha_1 + k)s'_1 + (\alpha_2 - k)s'_2 \in q_k(S)$ and since $k' - k > 0$ and $s'_1 - s'_2 > 0$, we have $(\alpha_1 + k)s'_1 + (\alpha_2 - k)s'_2 < t_k$. Now t_k is the smallest element in $q_k(S)$ not divisible by d, so d divides $(\alpha_1 + k)s'_1 + (\alpha_2 - k)s'_2$. We have $(\alpha_1 + k)s'_1 + (\alpha_2 - k)s'_2 = (\alpha_1 + k)(s'_1 - s'_2) + (\alpha_1 + k + \alpha_2 - k)s'_2 = (\alpha_1 + k)(s'_1 - s'_2) + ds'_2$. By assumption $\gcd(d, \alpha_1 + k) = 1$, so that d divides $s'_1 - s'_2$. But then d divides t_k, however by definition d does not divide t_k, and we have a contradiction. Therefore $t_k \notin q_{k'}(S)$ for $k' > k$, implying that $q_k(S) \neq q_{k'}(S)$ and the result follows. □

9 Composition of patterns

Let I, J and K be three ideals of three numerical semigroups. Also, for $i \in 1, \ldots, n'$, let $q_i : \mathcal{S}_n(I) \to J$ be a pattern sending non-increasing sequences of length n of elements in I to elements in J and let $p : \mathcal{S}_{n'}(J) \to K$ be a pattern sending non-increasing sequences of length n' of elements in J to elements in K. Define the polynomial composition of the patterns p and $q_1, \ldots, q_{n'}$ as $p \circ (q_1, \ldots, q_{n'}) = p(q_1(X_1, \ldots, X_n), \ldots, q_{n'}(X_1, \ldots, X_n))$.

Polynomial composition of patterns requires more than composition of polynomials for being well-defined.

Lemma 42. *The composition* $p\circ(q_1,\dots,q'_n)$ *of the patterns* $p:\mathcal{S}_{n'}(J)\to K$ *and* $q_1,\dots,q'_n:\mathcal{S}_n(I)\to J$ *is well-defined if the image of the* q_i *is contained in the domain of* p *and* $q_1(s_1,\dots,s_n)\geq\cdots\geq q_{n'}(s_1,\dots,s_n)$ *for any non-increasing sequence* $(s_1,\dots,s_n)\in\mathcal{S}_n(I)$.

Proof. Clear from the definition of pattern. □

Example 43. *If* S *is Arf, then* S *admits* $p_A(X_1,X_2,X_3)=X_1+X_2-X_3$. *From Lemma 42 we know that if* p *is a pattern of length* n *admitted by* S *and* $q_1,\dots,q_n$ *are patterns of length* n *admitted by* S *then* $p\circ(q_1,\dots,q_n)$ *is also admitted by* S, *whenever that composition is well-defined. Therefore, since* Y_1 *and* Y_2 *are patterns, and assuming* $Y_1\geq Y_2$, *we can make the change of variables* $X_1=X_2=Y_1$ *and* $X_3=Y_2$ *and we see that* $q(X_1,X_2)=p_A(X_1,X_1,X_2)=2X_1-X_2$ *is admitted by* S. *In other words,* $p_A(X_1,X_2,X_3)$ *induces* $q(X_1,X_2)$. *Actually, as was proved in [7], it turns out that* q *also induces* p_A, *and so* q *and* p_A *are equivalent.*

With other changes of variables we can obtain for example, with $a,b,c\in\mathbb{Z}$ *and* $a\geq b\geq c\geq 0$,

- $X_1=aY_1+aY_2$
- $X_2=bY_1+bY_2-bY_3$
- $X_3=cY_3$

we get $p_A(X_1,X_2,X_3)=(aY_1+aY_2)+(bY_1+bY_2-bY_3)-cY_3=(a+b)Y_1+(a+b)Y_2-(b+c)Y_3)$. *With* $a=b=1$ *that gives* $p_A(X_1,X_2,X_3)=2p_A(Y_1,Y_2,Y_3)$ *and with* $a=2$, $b=c=1$ *it gives* $p_A(X_1,X_2,X_3)=3Y_1+3Y_2-2Y_3$. *If instead, with* $a,b,c\in\mathbb{Z}$, $a\geq b\geq c\geq 0$ *we define*

- $X_1=aY_1+aY_2-aY_3$
- $X_2=bY_1+bY_2-bY_3$
- $X_3=cY_3$

then we get $p_A(X_1,X_2,X_3)=(aY_1+aY_2-aY_3)+(bY_1+bY_2-bY_3)-cY_3=(a+b)Y_1+(a+b)Y_2-(a+b+c)Y_3$ *which with* $a=b=c=1$ *gives* $p_A(X_1,X_2,X_3)=2Y_1+2Y_2-3Y_3$.

Acknowledgements

My gratitute goes to Ralf Fröberg and Christian Gottlieb without whom this article would not have been written, and for their extensive and invaluable help during its elaboration. I also want to thank Pedro García-Sánchez for pointing out that the patterns in Lemma 12, for which the partial sums of the coefficients

are larger than one, are exactly the strongly admissible patterns. The author acknowledges partial support from the Spanish MEC project ICWT (TIN2012-32757) and ARES (CONSOLIDER INGENIO 2010 CSD2007-00004).

References

[1] T.M. Apostol. Calculus Vol. 1 (second edition). Blaisdell Publ.Comp. 1967.

[2] V. Barucci, D.E. Dobbs and M. Fontana. Maximality Properties in Numerical Semigroups and Applications to One-Dimensional Analytically Irreducible Local Domains. Memoirs of the American Mathematical Society 598, 125, AMS, 1997.

[3] M. Bras-Amorós, A. Vico-Oton. On the Geil-Matsumoto Bound and the Length of AG codes, Designs, Codes and Cryptography, 70:1-2, (2014) pp. 117-125.

[4] M. Bras-Amorós and P.A. García-Sánchez. Patterns on numerical semigroups, Linear Algebra and its Applications, 414, (2006) pp. 652-669.

[5] M. Bras-Amorós, P.A. García-Sánchez, A. Vico-Oton. Non-homogeneous patterns on numerical semigroups, International Journal of Algebra and Computation, 23:6, (2013) pp. 1469.

[6] M. Bras-Amorós and K. Stokes. The semigroup of combinatorial configurations. Semigroup Forum, Springer, New York, 84;1, (2012) pp. 91–96.

[7] A. Campillo, J.I. Farrán, and C. Munuera. On the parameters of algebraic-geometry codes related to Arf semigroups. IEEE Transactions on information theory 46:7, (2000) pp. 2634–2638.

[8] J. Lipman. Stable ideals and Arf rings, Amer. J. Math. 93 (1971) 649-685.

[9] A. M. Robles-Pérez and J. C. Rosales. Frobenius pseudo-varieties in numerical semigroups. Annali di Matematica Pura ed Applicata, Online publication date: 21-Sep-2013.

[10] J.C. Rosales, M.B. Branco and D. Torrão. The Frobenius problem for Mersenne numerical semigroups. Manuscript.

[11] J.C. Rosales and P.A. García-Sánchez. Every numerical semigroup is one half of infinitely many symmetric numerical semigroups. Communications in Algebra 36:8 (2008) pp. 29102916.

[12] J.C. Rosales and P.A. García-Sánchez. Numerical semigroups, Springer, 2009.

[13] K. Stokes and M. Bras-Amorós. Linear non-homogeneous patterns and prime power generators in numerical semigroups associated to combinatorial configurations, Semigroup Forum 88:1, (2014) pp. 11–20.

[14] K. Stokes and M. Bras-Amorós. Patterns in semigroups associated with combinatorial configurations. In M. Izquierdo, S.A. Broughton, A.F. Costa, R.E. Rodríguez (Eds), "Riemann and Klein Surfaces, Automorphisms, Symmetries and Moduli Spaces", Contemporary Mathematics 629, American Mathematical Society, Providence, RI, 2014, 323–333.

[15] I. Swanson, Every numerical semigroup is one over d of infinitely many symmetric numerical semigroups, Commutative Algebra and its Applications (Fez), Morocco, (2008), pp. 23-28.